

# A ‘Super-Critical’ Nonlinear Wave Equation in 2 Space Dimensions

Michael Struwe

**Abstract.** These notes are an extended exposition of lectures given at the conference “*Nonlinear Analysis*”, Verbania, Sept. 25–29, 2010, where we reviewed the results from [11] on global well-posedness of the Cauchy problem for wave equations with exponential nonlinearities in 2 space dimensions for smooth, arbitrarily large radially symmetric data.

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## 1. Introduction

Consider the initial value problem for the equation

$$u_{tt} - \Delta u + ue^{u^2} = 0 \text{ on } \mathbb{R} \times \mathbb{R}^2 \quad (1)$$

with smooth Cauchy data

$$(u, u_t)|_{t=0} = (u_0, u_1) \in C^\infty(\mathbb{R}^2). \quad (2)$$

It suffices to consider times  $t > 0$ . Multiplying (1) with  $u_t$ , for a smooth global solution  $u$  of (1) we obtain the conservation law

$$0 = \frac{d}{dt} e(u) - \operatorname{div}(\nabla u \cdot u_t) \quad (3)$$

for the energy density

$$e(u) = \frac{1}{2}(|u_t|^2 + |\nabla u|^2 + e^{u^2} - 1). \quad (4)$$

and the density of momentum

$$m(u) = \nabla u \cdot u_t.$$

Since clearly  $|m(u)| \leq e(u)$ , integration of (3) over a truncated light cone yields

$$E(u(t), B_R(x_0)) := \int_{B_R(x_0)} e(u(t)) dx \leq E(u(s+t), B_{R+|s|}(x_0)) \quad (5)$$

for any  $x_0 \in \mathbb{R}^2$ ,  $R > 0$ , and  $0 < s+t, t$ . In particular, energy will spread with speed at most 1 and solutions for compactly supported data will have compact support at any given time  $t > 0$ . Integrating (3) over  $[0, t] \times \mathbb{R}^2$  then we find the global energy identity

$$E(u(t)) = \int_{\mathbb{R}^2} e(u(t)) dx = E(u(0)) \quad (6)$$

for all  $t > 0$ .

Equation (1) is related to the critical Sobolev embedding in 2 space dimensions. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ . Recall the Moser-Trudinger inequality

$$\sup_{u \in H_0^1(\Omega); \|\nabla u\|_{L^2(\Omega)}^2 \leq 1} \int_{\Omega} e^{4\pi u^2} dx < \infty; \quad (7)$$

see [6], [14]. The exponent  $\alpha = 4\pi$  is critical for this Orlicz space embedding in the sense that for any  $\alpha > 4\pi$  there holds

$$\sup_{u \in H_0^1(\Omega); \|\nabla u\|_{L^2(\Omega)}^2 \leq 1} \int_{\Omega} e^{\alpha u^2} dx = \infty. \quad (8)$$

On account of the obvious scaling property

$$\sup_{u \in H_0^1(\Omega); \|\nabla u\|_{L^2(\Omega)}^2 = 1} \int_{\Omega} e^{\alpha u^2} dx = \sup_{u \in H_0^1(\Omega); \|\nabla u\|_{L^2(\Omega)}^2 = \alpha} \int_{\Omega} e^{u^2} dx \quad (9)$$

and in view of (7), (8) the Cauchy problem for (1) with initial energy  $E(u(0)) < 2\pi$  then may be regarded as “*sub-critical*”, while the case  $E(u(0)) = 2\pi$  appears to be “*critical*”. In [2] Ibrahim, Majdoub, and Masmoudi demonstrated that problem (1), (2) is well-posed in these cases. Jointly with Nakanishi the same authors later even showed scattering for (1) whenever the initial energy  $E(u(0)) \leq 2\pi$ ; see [4].

On the other hand, in view of (8) above for  $E(u(0)) > 2\pi$  not even a locally uniform spatial  $L^1$ -bound is available for the term  $ue^{u^2}$ . In analogy with nonlinear wave equations

$$u_{tt} - \Delta u + u|u|^{p-2} = 0 \text{ on } \mathbb{R} \times \mathbb{R}^n$$

with  $p > \frac{2n}{n-2}$  in  $n \geq 3$  space dimensions, where the nonlinear term cannot be bounded in the dual space of  $H^1$  in terms of the Dirichlet energy, the Cauchy problem for equation (1) was therefore termed “*super-critical*” for initial data with energy  $E(u(0)) > 2\pi$ . The recent results [1], [3] of Ibrahim, Jrad, Majdoub, and Masmoudi, showing that the local solution of the Cauchy problem (1), (2) does not depend on the initial data in a locally uniformly continuous fashion when  $E(u(0)) > 2\pi$ , seemed to further justify this classification and sparked further interest in this equation, in the hope that the study of (1), (2) for large data might lead to progress on the challenging issues surrounding super-critical problems in general.

However, we shall see that problem (1), (2) never admits solutions that blow up in finite time, no matter how large the energy of the initial data is. Indeed, in [11] we proved the following result.

**Theorem 1.1.** *For any radially symmetric data  $(u_0, u_1) = (u_0(|x|), u_1(|x|)) \in C^\infty(\mathbb{R}^2)$  there exists a unique, smooth solution  $u = u(t, |x|)$  to the Cauchy problem (1), (2), defined for all time.*

In a recent paper [12], moreover, we were able to remove the assumption of radial symmetry of the data. No weighted energy estimates are required in the proof, as would be expected in a truly “super-critical” context. Whereas these results suggest that problem (1), (2) still belongs to the realm of “critical” equations, so far nothing is known about scattering for large energies or about continuous dependence in the energy norm, and there might still be surprises. These questions certainly will require further study. See [3], [5], [10], [13] for recent results on supercritical wave equations, and [7], [9] for further background.

In these lectures we present the detailed argument for regularity in the radial case. Indeed, this is the case where singularities would seem most likely to occur. In the following sections we first briefly review the local existence theory for problem (1), (2) for arbitrary smooth initial data and recall the global well-posedness results of [2] for small energy. Assuming that the local solution to (1), (2) for certain data cannot be extended to all of space-time, in Section 3 we arrive at a simplified blow-up scenario with a smooth, radially symmetric solution on a light cone blowing up at the origin. The blow-up is characterized by concentration of energy. In order to rule out blow-up, it thus suffices to show that, in fact, contrary to the above, the energy of our solution decays near the origin. As a first step towards this goal, by making full use of the energy inequality, in Section 4 we derive decay of the energy flux on the lateral boundary of the light cone. In view of radial symmetry, the decay of the flux implies pointwise estimates away from the axis  $x = 0$  which, when coupled with an argument of Shatah-Tahvildar-Zadeh [8], in Sections 5-7 allow to prove full energy decay. This concludes the proof. It is only in Section 5 where the assumption of radial symmetry crucially enters.

## 2. Local existence

In this section we allow arbitrary smooth initial data  $(u_0, u_1)$  for (1), (2).

**Lemma 2.1.** *Given smooth initial data  $(u_0, u_1)$ , there exists a smooth local solution  $u$  of the initial value problem (1), (2) in an open neighborhood of  $\{0\} \times \mathbb{R}^2$ .*

*Proof.* i) Suppose that  $(u_0, u_1)$  are compactly supported. Given  $L > 1$ , define the function  $g_L(u) = u \min\{e^{L^2}, e^{u^2}\}$  with potential  $G_L$  satisfying

$$2G_L(u) = (1 + (u^2 - L^2)_+)e^{u^2 - (u^2 - L^2)_+} \leq e^{u^2},$$

where  $s_+ = \max\{0, s\}$  for  $s \in \mathbb{R}$ . For any  $L > 1$  then  $g_L$  is uniformly Lipschitz. Hence the initial value problem

$$u_{tt} - \Delta u + g_L(u) = 0 \text{ on } \mathbb{R} \times \mathbb{R}^2 \quad (10)$$

with data (2) admits a unique global solution  $u = u_L$  of class  $H^2$  satisfying the energy estimate

$$E_L(u(t), B_R(x_0)) := \int_{B_R(x_0)} e_L(u(t)) dx \leq E_L(u(0), B_{R+t}(x_0)) \leq E(u(0)) \quad (11)$$

for any  $x_0 \in \mathbb{R}^2$ ,  $R > 0$ ,  $t > 0$ , where

$$e_L(u) = \frac{1}{2}(|u_t|^2 + |\nabla u|^2) + G_L(u) \leq e(u);$$

confer for example [9], pp. 59-61. In addition, upon taking spatial difference quotients

$$\partial_i^{(h)} u(t, x) = \frac{u(t, x + he_i) - u(t, x)}{h}$$

in (10), multiplying by  $\partial_t \partial_i^{(h)} u(t, x)$  and passing to the limit  $h \rightarrow 0$ , similar to (3) we have

$$\frac{1}{2} \frac{d}{dt} |D \nabla u|^2 - \operatorname{div}(\nabla^2 u \cdot \nabla u_t) \leq C_L |\nabla u| |D \nabla u| \text{ on } \mathbb{R} \times \mathbb{R}^2, \quad (12)$$

where  $D = (\frac{d}{dt}, \nabla)$  and where  $C_L$  is the Lipschitz constant of  $g_L$ . In particular, letting  $K_0^{t_0}$  be the light cone with base  $B_R(x_0)$ , truncated at height  $t_0$ , upon integration we find

$$\begin{aligned} & \sup_{0 < t < t_0} \int_{B_{R-t}(x_0)} |D \nabla u(t)|^2 dx \\ & \leq \int_{B_R(x_0)} |D \nabla u(0)|^2 dx + 2C_L \int_{K_0^{t_0}} |\nabla u| |D \nabla u| dx dt. \end{aligned}$$

Using Hölder's and Young's inequalities to estimate

$$\begin{aligned} \int_{K_0^{t_0}} |\nabla u| |D \nabla u| dx dt & \leq \left( t_0 \sup_{0 < t < t_0} \int_{B_{R-t}(x_0)} |D \nabla u|^2 dx \cdot \int_{K_0^{t_0}} |\nabla u|^2 dx dt \right)^{1/2} \\ & \leq \frac{1}{4C_L} \sup_{0 < t < t_0} \int_{B_{R-t}(x_0)} |D \nabla u(t)|^2 dx + C_L t_0 \int_{K_0^{t_0}} |\nabla u|^2 dx dt, \end{aligned}$$

then for any  $x_0 \in \mathbb{R}^2$ ,  $0 < t_0 < R$  we obtain

$$\sup_{0 < t < t_0} \int_{B_{R-t}(x_0)} |D \nabla u(t)|^2 dx \leq 2 \int_{B_R(x_0)} |D \nabla u(0)|^2 dx + 8C_L^2 t_0^2 E(u(0)).$$

By Sobolev's embedding  $H^2 \hookrightarrow L^\infty$  on any ball  $B_{R-t}(x_0)$  then, in particular,  $u$  is uniformly bounded on  $K_0^{t_0}$ . Choosing  $L > 1$  sufficiently large and then choosing  $t_0 > 0$  sufficiently small, we can achieve that  $|u_L(t, x)| < L$  for all  $(t, x) \in K_0^{t_0}$ ; that is,  $u = u_L$  solves the original problem (1), (2) on  $K_0^{t_0}$ . Since finitely many such truncated cones cover the support of  $u_0$  and  $u_1$ , and since by (11) the support of  $u_L(t)$  for any  $L > 1$  grows with speed at most 1, for sufficiently large  $L > 1$  and

sufficiently small  $t_0 > 0$  then  $u = u_L$  solves the original problem (1), (2) on all of  $[0, t_0] \times \mathbb{R}^2$ .

ii) For compactly supported smooth data  $(u_0, u_1), (v_0, v_1)$ , respectively, let  $u, v$ , respectively, be the local smooth solutions to (1), (2) obtained in step i). Suppose that  $u_0 = v_0, u_1 = v_1$  on  $B_R(x_0)$ . Then  $w = u - v$  satisfies the equation

$$w_{tt} - \Delta w + w(e^{u^2} - e^{v^2})/(u - v) = 0 \text{ on } \mathbb{R} \times \mathbb{R}^2 \quad (13)$$

with data  $w = 0, w_t = 0$  on  $B_R(x_0)$ . Multiplying (13) by  $w_t$  and noting that for small time  $0 < t < t_0 \leq R$  with a uniform constant  $C_0 > 0$  we can estimate  $|e^{u^2} - e^{v^2}| \leq C_0|u - v|$  on  $[0, t_0] \times B_R(x_0)$ , we obtain the conservation law

$$\frac{1}{2} \frac{d}{dt} |Dw|^2 - \operatorname{div}(\nabla w \cdot w_t) \leq C_0 |w| |w_t| \text{ on } [0, t_0] \times B_R(x_0).$$

Letting  $K_0^{t_0}$  be as in step i) above and recalling that  $Dw(0) = 0$  on  $B_R(x_0)$ , upon integration and by use of Hölder’s and Young’s inequalities as above we find

$$\begin{aligned} \sup_{0 < t < t_0} \int_{B_{R-t}(x_0)} |Dw(t)|^2 dx &\leq 2C_0 \int_{K_0^{t_0}} |w| |Dw| dx dt \\ &\leq \frac{1}{2} \sup_{0 < t < t_0} \int_{B_{R-t}(x_0)} |Dw(t)|^2 dx + 2C_0^2 t_0 \int_{K_0^{t_0}} |w|^2 dx dt. \end{aligned}$$

Finally, Poincaré’s inequality gives

$$\int_{K_0^{t_0}} |w|^2 dx dt \leq t_0^2 \int_{K_0^{t_0}} |w_t|^2 dx dt \leq t_0^3 \sup_{0 < t < t_0} \int_{B_{R-t}(x_0)} |Dw(t)|^2 dx.$$

Hence for sufficiently small  $t_0 > 0$  we have  $w = 0$  and therefore  $u = v$  on  $K_0^{t_0}$ .

iii) Given arbitrary smooth initial data  $(u_0, u_1)$  and any point  $x_0 \in \mathbb{R}^2$ , any radius  $R > 0$ , we may truncate the data outside the ball  $B_R(0)$  to obtain initial data with compact support. For sufficiently small  $t_0 > 0$ , possibly depending on  $x_0$  and  $R$ , the local solution  $v$  of (1), (2) for these data then provides a unique solution  $u$  to (1), (2) for the given data  $(u_0, u_1)$  in the truncated light cone  $K_0^{t_0}$  defined above. Thus for arbitrary smooth initial data  $(u_0, u_1)$  there always exists a local smooth solution  $u$  of (1), (2) in an open neighborhood of  $\{0\} \times \mathbb{R}^2$ .  $\square$

### 3. Simplified blow-up scenario

In preparing for the proof of Theorem 1.1 we now assume that the local solution  $u$  to (1), (2) for certain Cauchy data  $u_0, u_1 \in C^\infty(\mathbb{R}^2)$  cannot be smoothly extended to a neighborhood of some point  $(T_0, x_0)$  where  $T_0 \geq 0$ . We contend that we may assume that  $(u_0, u_1)$  are compactly supported,  $T_0 > 0$ , and that  $u \in C^\infty([0, T_0] \times \mathbb{R}^2)$ . Indeed, choose a function  $\tau = \tau(|x|) \in C_0^\infty(\mathbb{R}^2)$  such that  $\tau \equiv 1$  on a ball  $B_R(x_0)$  for some  $R > T_0$ . By Lemma 2.1 there exists a unique local solution  $\tilde{u}$  to (1), (2) with Cauchy data  $(\tau u_0, \tau u_1)$  and  $\tilde{u} \in C^\infty([0, T_1] \times \mathbb{R}^2)$  for some maximal number  $0 < T_1 \leq \infty$ . Moreover, as shown in step ii) of the proof of Lemma 2.1, the functions

$u$  and  $\tilde{u}$  agree on the truncated light cone  $K_0^{t_0}$ , where  $t_0 = \min\{T_0, T_1\}$ . It follows that  $T_1 \leq T_0$ ; otherwise  $\tilde{u}$  would yield a smooth extension of  $u$  in a neighborhood of  $(T_0, x_0)$ .

Note that we can characterize blow-up through concentration of energy. Indeed, from [2] we have the following criterion for a first blow-up point  $(T_0, x_0)$ .

**Lemma 3.1.** *There exists  $\varepsilon_0 > 0$  such that for any solution  $u \in C^\infty([0, T_0] \times \mathbb{R}^2)$  of (1) that cannot be smoothly extended to a neighborhood of a point  $(T_0, x_0)$  there holds*

$$E(u(t), B_{T_0-t}(x_0)) \geq \varepsilon_0 \text{ for all } 0 < t < T_0.$$

In fact, we may take  $\varepsilon_0 = \pi/10$ .

*Proof.* Suppose we have  $E(u(T), B_{T_0-T}(x_0)) < \pi/10$  for some  $0 < T < T_0$ . We may then choose  $R > 0$  such that  $E(u(T), B_{T_0-T+R}(x_0)) \leq \pi/10$ , and by (5) there holds the uniform energy bound

$$E(u(t), B_{T_0-t+R}(x_0)) \leq \pi/10 \text{ for all } T \leq t < T_0. \quad (14)$$

We claim that as a consequence of (14) there exists a number  $\rho > 0$  such that the functions  $e^{u^2(t)}$  are uniformly bounded in  $L^8(B_{T_0-t+\rho}(x_0))$  for  $T \leq t < T_0$ . To see this, it suffices to establish such a uniform bound on  $B_{2\rho}(x_1)$  for any such  $t$  and any point  $x_1 \in B_{T_0-t}(x_0)$ . To simplify the notation we shift coordinates so that  $x_1 = 0$ .

Observe that for any  $a > 0$ ,  $b > 1$  we can estimate

$$ab \leq (e^a - 1) + b \log b;$$

thus, for any  $a, b > 0$  there holds

$$ab \leq (e^a - 1) + b \log(1 + b).$$

For any Lipschitz continuous cut-off function  $0 \leq \tau \leq 1$  supported in  $B_R(0)$ , upon letting  $a = u^2(t)$ ,  $b = |\nabla \tau|^2$  in the previous estimate for any  $T \leq t < T_0$  we can therefore bound

$$\begin{aligned} \|\nabla(\tau u(t))\|_{L^2(B_R(0))}^2 &\leq 2\|\nabla u(t)\|_{L^2(B_R(0))}^2 + 2\|u(t)\nabla \tau\|_{L^2(B_R(0))}^2 \\ &= 2 \int_{B_R(0)} (|\nabla u(t)|^2 + u^2(t)|\nabla \tau|^2) dx \\ &\leq 4E(u(t), B_R(0)) + 2 \int_{B_R(0)} |\nabla \tau|^2 \log(1 + |\nabla \tau|^2) dx. \end{aligned}$$

Choosing  $\tau = \tau_L(|x|)$  with  $\tau_L(r) = \min\{1, (\log \log \log(\frac{1}{r}) - \log \log \log L)_+\}$ , for sufficiently large  $L > 1/R$  we can achieve that

$$2 \int_{B_R(0)} |\nabla \tau|^2 \log(1 + |\nabla \tau|^2) dx \leq C \int_0^{1/L} \frac{\log(1/r) dr}{(\log \log(1/r) \log(1/r))^2 r} \leq \frac{\pi}{10}.$$

For such  $L$  then  $\tau_L u(t) \in H_0^1(B_R(0))$  satisfies  $\|\nabla(\tau_L u(t))\|_{L^2(B_R(0))}^2 \leq \pi/2$  and the Moser-Trudinger inequality implies a uniform bound for  $e^{\tau_L^2 u^2(t)}$  in  $L^8(B_R(0))$ .

Choosing  $\rho > 0$  such that

$$\log \log \log \left( \frac{1}{2\rho} \right) = 1 + \log \log \log L$$

then  $\tau = 1$  on  $B_{2\rho}(0)$ , and our claim follows.

Again let  $D = (\frac{d}{dt}, \nabla)$  denote the space-time differential. Differentiating (1) in space and multiplying by  $\nabla u_t$ , similar to (12) we obtain the identity

$$\frac{d}{dt} |D\nabla u|^2 - \operatorname{div}(\nabla^2 u_t \cdot \nabla u_t) = \nabla u(1 + 2u^2)e^{u^2} \nabla u_t. \quad (15)$$

By Young’s inequality and the obvious estimate  $1 + u^2 \leq e^{u^2}$  we can estimate the right hand side

$$|\nabla u(1 + 2u^2)e^{u^2} \nabla u_t| \leq C e^{8u^2} + |\nabla u|^4 + |D\nabla u|^2,$$

where  $C$  is an absolute constant. For any  $T \leq t < T_0$ , with  $D(t) = \{t\} \times B_{T_0-t+\rho}(x_0)$  an interpolation inequality of Gagliardo-Nirenberg-Ladyzhenskaya allows to bound

$$\begin{aligned} \int_{D(t)} |\nabla u|^4 dx &\leq C \int_{D(t)} |\nabla u|^2 dx \cdot \int_{D(t)} (\rho^{-2} |\nabla u|^2 + |\nabla^2 u|^2) dx \\ &\leq C E(u(t), D(t)) \left( \rho^{-2} E(u(t), D(t)) + \int_{D(t)} |\nabla^2 u|^2 dx \right). \end{aligned}$$

In view of the energy inequality (5) and the fact that the functions  $e^{u^2(t)}$  are uniformly bounded in  $L^8(D(t))$  for  $T \leq t < T_0$ , hence we find

$$\int_{D(t)} |\nabla u(1 + 2u^2)e^{u^2} \nabla u_t| dx \leq C + C \int_{D(t)} |D\nabla u|^2 dx.$$

Upon integrating the conservation law (15) over the truncated backward light cone

$$K_T^{T_0} = \{(t, x); T < t \leq T_0, |x - x_0| < T_0 - t + \rho\}$$

there results a uniform bound for  $u(t)$  in  $H^2(D(t))$ ,  $T < t \leq T_0$ . By Sobolev’s embedding then  $u \in L^\infty(K_T^{T_0})$  and  $u$  smoothly extends to a neighborhood of  $(T_0, x_0)$ , contrary to assumption.  $\square$

So far, our discussion has been completely general. In the radial case now, a first singularity can only occur at  $x_0 = 0$ . Indeed, by the energy inequality and Sobolev’s embedding the solution  $u$  is uniformly bounded away from the axis  $x = 0$ . Hence if  $u$  is smooth on  $[0, T_0[ \times \mathbb{R}^2$  and  $x_0 \neq 0$ ,  $u$  extends smoothly to a neighborhood of  $(T_0, x_0)$ . Alternatively, we may also invoke Lemma 3.1 to see this: If  $(T_0, x_0)$  is a first singularity and  $x_0 \neq 0$ , then any point  $(T_0, x_1)$  with  $|x_1| = |x_0|$  also is a first singularity and by Lemma 3.1 for any  $0 < t < T_0$  we have  $E(u(t), B_{T_0-t}(x_1)) \geq \varepsilon_0$ . But for any  $K \in \mathbb{N}$  and sufficiently large  $t < T_0$  we can find points  $x_k$  with  $|x_k| = |x_0|$ ,  $1 \leq k \leq K$ , such that the balls  $B_{T_0-t}(x_k)_{1 \leq k \leq K}$  are disjoint, and we conclude

$$E(u(t)) \geq \sum_{k=1}^K E(u(t), B_{T_0-t}(x_k)) \geq K\varepsilon_1,$$

contradicting the energy estimate (5) for large  $K$ .

Shifting time by  $T_0$  and then reversing the arrow of time, in the following we may therefore assume that we have a compactly supported solution  $u = u(t, |x|) \in C^\infty([0, T_0] \times \mathbb{R}^2)$  of (1) blowing up at  $(0, 0)$ .

#### 4. Improved energy inequality and flux decay

To proceed we will need a sharper version of the local energy estimate (5). For ease of notation we state this estimate only in the case when  $x_0 = 0$  which will be the only case of interest later. Moreover, we replace our original choice  $e^{u^2} - 1$  of potential for the nonlinear term in (1) by the function  $e^{u^2}$ . While the former choice has the advantage of attributing a finite total energy to any smooth compactly supported data, the latter one turns out to be more convenient when working on a bounded domain.

Denoting as  $v(y) = u(|y|, y)$  the restriction of  $u$  to the lateral boundary

$$M_S^T = \{z = (t, x); S \leq t \leq T, |x| = t\}$$

of the truncated forward light cone

$$K_S^T = \{z = (t, x); S \leq t \leq T, |x| \leq t\}$$

with vertex at  $z = (0, 0)$ , and letting

$$Flux(u, M_S^T) := \frac{1}{2} \int_{B_T \setminus B_S(0)} (|\nabla v|^2 + e^{v^2}) dy,$$

upon integrating (3) over  $K_S^T$  we find the identity

$$E(u(S), B_S(0)) + Flux(u, M_S^T) = E(u(T), B_T(0)) \quad (16)$$

for all  $0 < S < T \leq T_0$ . In particular,  $\lim_{T \downarrow 0} E(u(T), B_T(0))$  exists and we conclude decay of the flux

$$Flux(u, M_0^T) := \sup_{0 < S < T} Flux(u, M_S^T) \rightarrow 0 \text{ as } T \downarrow 0. \quad (17)$$

Finally, from (16) and Lemma 3.1 we also have the uniform bounds

$$0 < \varepsilon_0 \leq E(u(T), B_T(0)) \leq E(u(T_0), B_{T_0}(0)) \leq E(u(T_0)) =: E_0 \quad (18)$$

for  $0 < T < T_0$ . We also denote  $M^T = M_0^T$ ,  $K^T = K_0^T$  for brevity.

#### 5. Exterior energy decay

Our aim is to show that contrary to (18) the energy  $E(u(t), B_t(0)) \rightarrow 0$  as  $t \downarrow 0$ . The first crucial step in deriving this energy decay is the following result.



**Lemma 5.1.** *For any number  $0 < \lambda \leq 1$  there holds*

$$E(u(t), B_t(0) \setminus B_{\lambda t}(0)) \rightarrow 0 \text{ as } t \downarrow 0.$$

The proof of Lemma 5.1 has two ingredients.

### 5.1. Pointwise estimates

By radial symmetry we have  $v(y) = u(|y|, y) = v(|y|)$ . For  $0 < t < T_1 \leq T_0$  by Hölder’s inequality then we can bound

$$\begin{aligned} |v(t)| &\leq |v(T_1)| + \int_t^{T_1} |v'(s)| ds \leq |v(T_1)| + \left( \int_t^{T_1} |\nabla v|^2 s ds \cdot \int_t^{T_1} \frac{ds}{s} \right)^{1/2} \\ &\leq |v(T_1)| + Flux^{1/2}(u, M_t^{T_1}) \log^{1/2}(T_1/t). \end{aligned}$$

By (17) we may choose  $0 < T_1 \leq \min\{1, T_0\}$  such that for all  $0 < t \leq T_1$  there holds

$$Flux^{1/2}(u, M_t^{T_1}) \leq Flux^{1/2}(u, M^{T_1}) \leq 1/3.$$

Also fixing  $0 < T_2 \leq T_1$  so that  $6|v(T_1)| \leq \log^{1/2}(1/t)$  for  $0 < t \leq T_2$  and observing that  $\log(T_1/t) \leq \log(1/t)$  for our choice of  $T_1$ , we thus obtain the bound

$$|v(t)| \leq \frac{1}{2} \log^{1/2}(1/t) \text{ for all } 0 < t \leq T_2. \quad (19)$$

The estimate (19) extends into the interior of the light cone. Indeed, for any  $0 < \lambda \leq 1$  and any  $(t, x)$  with  $0 < \lambda t \leq |x| \leq t \leq T_2$  by Hölder’s inequality we can estimate

$$\begin{aligned} |u(t, |x|)| &\leq |u(t, t)| + \int_{|x|}^t |u_r(t, r)| dr \leq |v(t)| + \left( \int_{\lambda t}^t |\nabla u|^2 r dr \cdot \int_{\lambda t}^t \frac{dr}{r} \right)^{1/2} \\ &\leq \frac{1}{2} \log^{1/2}(1/t) + (E(u(t), B_t(0)))^{1/2} \log^{1/2}(1/\lambda) \\ &\leq \frac{1}{2} \log^{1/2}(1/t) + E_0^{1/2} \log^{1/2}(1/\lambda) \leq \log^{1/2}(1/t), \end{aligned} \quad (20)$$

provided that  $0 < t \leq T_3$  for suitable  $0 < T_3 = T_3(\lambda) \leq T_2$ . Observing that  $t \log(1/t) \leq 1$  for all  $t > 0$ , for  $0 < \lambda t \leq |x| \leq t \leq T_3$  then we obtain the bound

$$|x|^2 u^2(t, x) e^{u^2(t, x)} \leq t \log(1/t) \leq 1. \quad (21)$$

### 5.2. Propagation of flux decay

The pointwise estimate (21) above allows to use a method of Shatah and Tahvildar-Zadeh [8], Lemma 2.2, to propagate the decay of flux from the lateral boundary into the interior of the light cone. For completeness we give the proof in detail. Note that we succeed in simplifying a key part of the original argument in [8].

Set  $r = |x|$  and let

$$e = \frac{1}{2}(u_r^2 + u_t^2 + e^{u^2}), \quad m = u_r \cdot u_t.$$

Also letting

$$F(u) = e^{u^2}, \quad f(u) = 2ue^{u^2}, \quad L = \frac{1}{2}(u_r^2 - u_t^2 - F(u)) - rf(u)u_r,$$

we compute

$$\partial_t(re) - \partial_r(rm) = 0, \quad \partial_t(rm) - \partial_r(re) = L, \quad (22)$$

where the first equation corresponds to the conservation law (3).

Introduce the characteristic coordinates

$$\xi = t + r, \quad \eta = t - r,$$

and define non-negative quantities  $A, B$  with

$$A^2 = r(e - m), \quad B^2 = r(e + m).$$

Upon adding and subtracting the equations (22), respectively, we arrive at the identities

$$\partial_\xi A^2 = -L/2, \quad \partial_\eta B^2 = L/2. \quad (23)$$

Given any  $0 < \lambda \leq 1$  now the pointwise estimate (21) for any  $(t, r)$  with  $0 < \lambda t \leq r \leq t \leq T_3$  permits to bound

$$r^2 f^2(u) = 4r^2 u^2 e^{u^2} F(u) \leq 4F(u).$$

Thus for any such  $(t, r)$  we have

$$\begin{aligned} L^2 &= \frac{1}{4}(u_r^2 - u_t^2 - F(u))^2 - (u_r^2 - u_t^2 - F(u))rf(u)u_r + r^2 f^2(u)u_r^2 \\ &\leq (u_r^2 - u_t^2)^2 + F^2(u) + 2r^2 f^2(u)u_r^2 \leq (u_r^2 - u_t^2)^2 + F^2(u) + 8F(u)u_r^2 \\ &\leq 4((u_r^2 - u_t^2)^2 + 2(u_r^2 + u_t^2)F(u) + F(u)^2) \\ &= C(e^2 - m^2) = C\frac{A^2 B^2}{r^2}. \end{aligned} \quad (24)$$

Given  $0 < \varepsilon < 1$ , in view of (17) we may fix  $0 < \xi_0 =: 2T_\varepsilon < T_3$  so that

$$\int_0^{\xi_0} B^2(\xi, 0) d\xi = \pi^{-1} Flux(u, M^{T_\varepsilon}) < \varepsilon. \quad (25)$$

Note that by (16) we also have

$$\int_0^{\xi_0/2} A^2(\xi_0, \eta) d\eta \leq E_0.$$

By absolute continuity of the Lebesgue integral, given  $\varepsilon > 0$  there is a number  $0 < \eta_\varepsilon \leq \xi_0/2$  such that for any  $0 < \eta_1 < \eta_\varepsilon$  we have

$$\int_0^{\eta_1} A^2(\xi_0, \eta) d\eta < \varepsilon. \quad (26)$$

Given  $0 < \lambda < 1$ , for any  $\xi_1 > 0$  set  $\eta_1 = \frac{1-\lambda}{1+\lambda}\xi_1$ . Since  $\eta_1 \downarrow 0$  as  $\xi_1 \downarrow 0$ , we can find a number  $0 < \xi_\varepsilon \leq \xi_0/2$  such that for  $0 < \xi_1 < \xi_\varepsilon$  we have  $0 < \eta_1 < \eta_\varepsilon$  and (26) holds.

Fix  $0 < \xi_1 < \xi_\varepsilon$  with corresponding  $\eta_1$ . For any  $(\xi_2, \eta_2)$  with  $\xi_1 \leq \xi_2 \leq \xi_0$ ,  $0 \leq \eta_2 \leq \eta_1$ , upon integrating the equations (23) over the intervals  $[\xi_2, \xi_0]$ ,  $[0, \eta_2]$ , respectively, on account of (24)-(26) we obtain

$$\begin{aligned} & \int_0^{\eta_2} A^2(\xi_2, \eta) d\eta + \int_{\xi_2}^{\xi_0} B^2(\xi, \eta_2) d\xi \\ & \leq \int_0^{\eta_2} A^2(\xi_0, \eta) d\eta + \int_{\xi_2}^{\xi_0} B^2(\xi, 0) d\xi + \int_0^{\eta_2} \int_{\xi_2}^{\xi_0} |L| d\xi d\eta \\ & \leq 2\varepsilon + C \int_0^{\eta_2} \int_{\xi_2}^{\xi_0} \frac{AB}{r} d\xi d\eta. \end{aligned} \quad (27)$$

Note that with a constant  $C = C(\lambda)$  we can bound  $\xi \leq Cr$  throughout the region  $[\xi_1, \xi_0] \times [0, \eta_1]$ . Therefore we can estimate

$$\begin{aligned} \int_0^{\eta_2} \int_{\xi_2}^{\xi_0} \frac{AB}{r} d\xi d\eta & \leq C \int_0^{\eta_2} \int_{\xi_1}^{\xi_0} \frac{AB}{\xi} d\xi d\eta \\ & \leq C \left( \int_0^{\eta_2} \int_{\xi_1}^{\xi_0} \frac{A^2}{\xi^{3/2}} d\xi d\eta \cdot \int_0^{\eta_2} \int_{\xi_1}^{\xi_0} \frac{B^2}{\xi^{1/2}} d\xi d\eta \right)^{1/2} \\ & \leq C \left( \xi_1^{-1} \sup_{\xi_1 < \xi < \xi_0} \int_0^{\eta_2} A^2(\xi, \eta) d\eta \cdot \int_0^{\eta_2} \int_{\xi_1}^{\xi_0} B^2 d\xi d\eta \right)^{1/2}. \end{aligned} \quad (28)$$

Inserting (28) in (27) and using Young’s inequality, for all  $\xi_2 \in [\xi_1, \xi_0]$  we then find

$$\begin{aligned} & \int_0^{\eta_2} A^2(\xi_2, \eta) d\eta + \int_{\xi_2}^{\xi_0} B^2(\xi, \eta_2) d\xi \\ & \leq 2\varepsilon + \frac{1}{2} \sup_{\xi_1 < \xi < \xi_0} \int_0^{\eta_2} A^2(\xi, \eta) d\eta + C\xi_1^{-1} \int_0^{\eta_2} \int_{\xi_1}^{\xi_0} B^2 d\xi d\eta. \end{aligned} \quad (29)$$

Passing to the supremum with respect to  $\xi_2 \in [\xi_1, \xi_0]$ , we can absorb the second term on the right in the left hand side and we obtain the uniform bound

$$\sup_{\xi_1 < \xi < \xi_0} \int_0^{\eta_2} A^2(\xi, \eta) d\eta \leq 4\varepsilon + C\xi_1^{-1} \int_0^{\eta_2} \int_{\xi_1}^{\xi_0} B^2 d\xi d\eta \quad (30)$$

for any  $\eta_2 \in [0, \eta_1]$ . Inserting this bound in (29) and setting  $\xi_2 = \xi_1$ , with a uniform constant  $C_1$  for all  $\eta_2 \in [0, \eta_1]$  we find

$$\int_{\xi_1}^{\xi_0} B^2(\xi, \eta_2) d\xi \leq 4\varepsilon + C_1\xi_1^{-1} \int_0^{\eta_2} \int_{\xi_1}^{\xi_0} B^2 d\xi d\eta.$$

For the function

$$\Phi(\eta_2) := \int_0^{\eta_2} \int_{\xi_1}^{\xi_0} B^2 d\xi d\eta,$$

this translates into

$$\frac{d}{d\eta} \Phi \leq 4\varepsilon + C_1\xi_1^{-1} \Phi.$$

The function

$$f(\eta) = e^{-C_1\eta/\xi_1} \Phi(\eta)$$

then satisfies

$$\frac{d}{d\eta} f \leq 4\varepsilon e^{-C_1\eta/\xi_1} \leq 4\varepsilon, \quad f(0) = 0.$$

Hence

$$f(\eta_1) \leq 4\varepsilon\eta_1 \leq 4\varepsilon\xi_1.$$

It follows that

$$\Phi(\eta_1) \leq 4\varepsilon\xi_1 e^{C_1\eta_1/\xi_1} \leq C\varepsilon\xi_1,$$

and (30) implies

$$\int_0^{\eta_1} A^2(\xi_1, \eta) d\eta \leq 4\varepsilon + C\xi_1^{-1}\Phi(\eta_1) \leq C\varepsilon.$$

But then by energy conservation (3) – best written in the form (22) – at time  $t = \frac{\xi_1 + \eta_1}{2} = \frac{\xi_1}{1+\lambda}$  we have

$$E(u(t), B_t(0) \setminus B_{\lambda t}(0)) \leq \int_0^{\eta_1} A^2(\xi_1, \eta) d\eta + \int_{\xi_1}^{\xi_1 + \eta_1} B^2(\xi, 0) d\xi \leq C\varepsilon.$$

Since  $\varepsilon > 0$  and  $\xi_1 \in ]0, \xi_\varepsilon[$  are arbitrary, the proof of Lemma 5.1 is complete.  $\square$

## 6. Time derivative decay

By arguing like Shatah and Tahvildar-Zadeh [8], Corollary 2.3, from Lemma 5.1 we deduce the following result.

**Lemma 6.1.** *We have*

$$\frac{1}{T} \int_{K^T} |u_t|^2 dz \rightarrow 0 \text{ as } T \downarrow 0.$$

*Proof.* Multiplying (1) by  $x \cdot \nabla u$  we obtain the identity

$$0 = \frac{d}{dt} (u_t x \cdot \nabla u) + \operatorname{div} \left( \frac{x}{2} (|\nabla u|^2 - |u_t|^2 + e^{u^2}) - \nabla u x \cdot \nabla u \right) + |u_t|^2 - e^{u^2}.$$

Upon integrating this equation over  $K_S^T$  and letting  $S \rightarrow 0$ , we find

$$\begin{aligned} \int_{K^T} |u_t|^2 dz &\leq \int_{K^T} e^{u^2} dz - \int_{\{T\} \times B_T(0)} (u_t x \cdot \nabla u) dx + CT \operatorname{Flux}(u, M_0^T) \\ &\leq \int_{K^T} e^{u^2} dz + o(T), \end{aligned} \tag{31}$$

where  $o(T)/T \rightarrow 0$  as  $T \rightarrow 0$  on account of Lemma 5.1 and (17). Next we multiply (1) by  $u/\log(1/t)$  to obtain the identity

$$\begin{aligned} 0 = \frac{d}{dt} \left( \frac{u_t u}{\log(1/t)} \right) - \operatorname{div} \left( \frac{u \nabla u}{\log(1/t)} \right) + \frac{|\nabla u|^2 - |u_t|^2}{\log(1/t)} \\ - \frac{u_t u}{t \log^2(1/t)} + \frac{u^2 e^{u^2}}{\log(1/t)}. \end{aligned} \tag{32}$$

Estimating  $|u| \leq |u - v| + |v|$ , by Poincaré’s inequality, (18) and (19) for  $0 < t \leq T \leq T_2$  we can bound

$$\begin{aligned} \int_{\{t\} \times B_t(0)} |u_t u| dx &\leq C \left( \int_{\{t\} \times B_t(0)} |u_t|^2 dx \cdot \int_{\{t\} \times B_t(0)} (|u - v|^2 + |v|^2) dx \right)^{1/2} \\ &\leq Ct \left( \int_{\{t\} \times B_t(0)} |\nabla u|^2 dx + v^2(t) \right)^{1/2} \leq CT \log^{1/2}(1/T); \end{aligned}$$

therefore

$$\int_{\{T\} \times B_T(0)} \frac{|u_t u|}{\log(1/T)} dx = o(T), \quad \int_{K^T} \frac{|u_t u|}{t \log^2(1/t)} dz \leq C \frac{T}{\log^{1/2}(1/T)} = o(T).$$

Bounding the remaining terms in (32) in similar fashion, we finally obtain that

$$\int_{K^T} \frac{u^2 e^{u^2}}{\log(1/t)} dz \leq o(T).$$

Together with (31) the latter estimate yields

$$\int_{K^T} |u_t|^2 dz \leq \int_{K^T} \left(1 - \frac{u^2}{\log(1/t)}\right) e^{u^2} dz + o(T) \leq o(T).$$

Here we observe that  $(1 - \frac{u^2}{\log(1/t)}) \leq 0$  unless  $u^2 \leq \log(1/t)$ ; therefore

$$\int_{K^T} \left(1 - \frac{u^2}{\log(1/t)}\right) e^{u^2} dz \leq \int_{K^T} \frac{1}{t} dz \leq CT^2. \quad \square$$

## 7. Energy decay: Proof of Theorem 1.1

By Lemma 6.1 there is a sequence of numbers  $T_k \downarrow 0$  as  $k \rightarrow \infty$  such that for  $t = T_k$  and  $t = T_k/2$  there holds

$$\int_{\{t\} \times B_t(0)} |u_t|^2 dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

For such  $T = T_k$  let

$$\bar{v} = \bar{v}_k = \oint_{B_T \setminus B_{T/2}(0)} v \, dy = \frac{4}{3\pi T^2} \int_{B_T \setminus B_{T/2}(0)} v \, dy.$$

Note that (19) implies that  $|\bar{v}| \leq \log^{1/2}(1/T)$  for large  $k$ . Multiply (1) by  $(u - \bar{v})$  to obtain the identity

$$0 = \frac{d}{dt} (u_t(u - \bar{v})) - \operatorname{div}(\nabla u(u - \bar{v})) + |\nabla u|^2 - |u_t|^2 + u(u - \bar{v})e^{u^2}. \quad (33)$$

Integrating (33) over  $K_{T/2}^T$  we then obtain

$$\begin{aligned} &\int_{K_{T/2}^T} (|\nabla u|^2 - |u_t|^2 + u(u - \bar{v})e^{u^2}) \, dz \\ &= \frac{1}{\sqrt{2}} \int_{M_{T/2}^T} (u_t + u_r)(u - \bar{v}) \, do - \int_{\{t\} \times B_t(0)} u_t(u - \bar{v}) \, dx \Big|_{t=T/2}^T. \end{aligned}$$

At  $t = T$  and  $t = T/2$  we estimate  $|u - \bar{v}| \leq |u - v(t)| + |v(t) - \bar{v}|$  and observe that by Poincaré's inequality we can bound

$$\int_{\{t\} \times B_t(0)} |u - v(t)|^2 dx \leq CT^2 \int_{\{t\} \times B_t(0)} |\nabla u|^2 dx \leq CT^2$$

as well as

$$|v(t) - \bar{v}|^2 \leq \int_{B_T \setminus B_{T/2}(0)} |v(t) - v|^2 dy \leq C \text{Flux}(u, M_{T/2}^T) \leq C. \quad (34)$$

Thus, at  $t = T$  and  $t = T/2$  we have

$$\begin{aligned} \int_{\{t\} \times B_t(0)} |u_t| |u - \bar{v}| dx &\leq C \left( \int_{\{t\} \times B_t(0)} |u_t|^2 dx \int_{\{t\} \times B_t(0)} |u - \bar{v}|^2 dx \right)^{1/2} \\ &\leq CT \left( \int_{\{t\} \times B_t(0)} |u_t|^2 dx \right)^{1/2} \left( \int_{\{t\} \times B_t(0)} |\nabla u|^2 dx + \text{Flux}(u, M_{T/2}^T) \right)^{1/2} = o(T). \end{aligned}$$

Similarly, by Hölder's inequality and (34) we find that

$$\begin{aligned} \frac{1}{\sqrt{2}} \int_{M_{T/2}^T} (u_t + u_r)(u - \bar{v}) do \\ \leq \int_{B_T \setminus B_{T/2}(0)} |\nabla v| |v - \bar{v}| dy \leq CT \text{Flux}(u, M_{T/2}^T) = o(T). \end{aligned}$$

Finally, recalling (18) and observing that the bound

$$u(u - \bar{v}) = |u - \bar{v}/2|^2 - |\bar{v}|^2/4 \geq -|\bar{v}|^2$$

may be improved to yield  $u(u - \bar{v}) \geq 1$  for  $|u(z)| \geq 1 + |\bar{v}|$ , with Lemma 6.1 we find

$$\begin{aligned} T\varepsilon_0 &\leq \int_{K_{T/2}^T} (|\nabla u|^2 + |u_t|^2 + e^{u^2}) dz \\ &\leq \int_{K_{T/2}^T} (|\nabla u|^2 - |u_t|^2 + u(u - \bar{v})e^{u^2}) dz \\ &\quad + 2 \int_{K_{T/2}^T} |u_t|^2 dz + \int_{\{z \in K_{T/2}^T; |u(z)| < 1 + |\bar{v}|\}} (1 + |\bar{v}|^2) e^{(1 + |\bar{v}|)^2} dz \\ &\leq \int_{K_{T/2}^T} (1 + \log(1/T)) e^{(1 + \log^{1/2}(1/T))^2} dz + o(T) \leq o(T). \end{aligned}$$

For large  $k$  a contradiction results, which ends the proof of Theorem 1.1.

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Michael Struwe  
Mathematik  
ETH-Zürich  
CH-8092 Zürich  
Switzerland

e-mail: [struwe@math.ethz.ch](mailto:struwe@math.ethz.ch)

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